UDC 532.5

ON THE EXISTENCE OF FLOW OF A HEAVY PERFECT FLUID IN A CHANNEL WITH A SLOPING BOTTOM

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Solvability of the problem of a flow of a heavy perfect fluid with free boundary in a channel with a bottom sloping without bounds is proved under the conditions that the Froude number is greater than unity.

Flows in a channel the bottom of which has two horizontal asymptotics, were studied earlier [1 —3] under analogous conditions. The problems in which a heavy fluid flows out of a vessel with the rate of flow increasing withour bounds, were studied for the arbitrary [4, 5] and for sufficiently large [6] values of the Froude number. However, in all the above problems the free boundary, in contrast to the flow in a channel, passes through a single point at infinity.

Let us consider, in the $z = x + iy$ plane, a steady potential flow of a perfect incompressible heavy fluid with a free boundary, in a channel the bottom of which consists of two rays emerging from the point $z = 0$ and forming the angles of π and $-\alpha\pi$ with the positive direction of the x-axis. The velocity vector at $x \to -\infty$ and the vector of acceleration due to gravity have the corresponding projections $(v_0, 0)$ and $(0, -\gamma)$. Let us introduce the following notation: v and θ are the modulus of the velocity vector and the angle between this vector and the x-axis, φ and ψ are the velocity potential and stream function, $w = \varphi + i\psi$, ψ_0 is the fluid flow rate, $\omega = \tau - i\theta = \ln (v_0^{-1}dw / dz)$, $\lambda = \gamma\psi_0 / v_0^3$.

Mapping the strip $0 \leqslant \psi \leqslant \psi_0$ in the w-variable plane onto the strip $0 \leqslant \eta \leqslant \pi / 2$ in the $\zeta = \xi + i\eta$ variable plane by means of the function $w = 2\psi_0\zeta / \pi$ we obtain, from the Bernoulli equation in the usual manner,

$$\exp \left[3\tau \left(\xi + i\frac{\pi}{2}\right)\right] = 1 - \frac{6\lambda}{\pi} \int\limits_{-\infty}^{\xi} \sin\theta \left(\xi + i\frac{\pi}{2}\right) d\xi \tag{1}$$

Function $\omega(\zeta)$ satisfies the boundary conditions (1) and the conditions

$$\omega(-\infty + i\eta) = 0 \quad (0 \leqslant \eta \leqslant \pi / 2), \quad \theta(\xi) = 0 \quad (\xi < 0), \quad \theta(\xi) = -\alpha\pi \quad (\xi > 0)$$

Using the Will's formulas we obtain

$$u(\xi) = D[g(\xi)] = D_1[g(\xi)] + D_2[g(\xi)] \tag{2}$$

$$D_k[g(\xi)] = -\frac{1}{\pi} \int\limits_{m_k}^{n_k} g(t) \ln \left| \operatorname{th} \frac{t - \xi}{2} \right| dt$$

$$k = 1, 2, \quad m_1 = -\infty, \quad m_2 = n_1 = \xi_0, \quad n_2 = \infty$$

$$(u(\xi) = -\theta(\xi + i\pi/2) - u_0(\xi), \quad u_0(\xi) = \alpha(\pi - 2\,\mathrm{arctg}\, e^{-\xi}),$$
$$g(\xi) = d\tau(\xi + i\pi/2)/d\xi$$

where ξ_0 is an arbitrary number. The operators D_k are positive.
From (1) and (2) we obtain

$$u(\xi) = D[G(\xi)] \tag{3}$$
$$G(\xi) = \frac{2\lambda}{\pi}\sin[u_0(\xi) + u(\xi)]\left[1 + \frac{6\lambda}{\pi}\int_{-\infty}^{\xi}\sin[u_0(\xi) + u(\xi)]\,d\xi\right]^{-1}$$

Equations (3) are equivalent to an operator equation of the type $u = T(u)$, and the solvability of the latter is proved below with help of the Schauder's principle.

Let $\xi_0 = \ln\sqrt{3}$ and let the parameters α and λ satisfy the conditions

$$0 < \alpha < 1, \quad 0 \leqslant \lambda < \min[1, (1-\alpha)\pi - \delta] \tag{4}$$

where $\delta > 0$ is arbitrarily small (the condition for λ is the same as the inequalities (2.28) and (4.31) in [1]). Let E be a space of functions continuous on $[-\infty, \infty]$ with the norm $\|u\| = \sup|u(\xi)|$, and $H = H(C_1, C_2, \beta)$ be a closed set of E the elements $u(\xi)$ of which satisfy the conditions

$$0 \leqslant u(\xi) \leqslant (1-\alpha)\pi - \delta \quad (|\xi| < \infty) \tag{5}$$

$$u(\xi) \leqslant C_1 e^{-\beta|\xi|} \ (\xi \leqslant \xi_0), \quad u(\xi) \leqslant C_2/\xi \quad (\xi \geqslant \xi_0) \tag{6}$$
$$0 < \beta < 1, \quad C_1 > 0, \quad C_2 > 0$$

We shall find the estimate of $F(\xi) = D[G(\xi)] = T(u)$ for $u \in H$. We note that

$$0 \leqslant u_0(\xi) \leqslant \alpha\pi \quad (|\xi| < \infty) \tag{7}$$

$$u_0(\xi) \leqslant 4\alpha e^{-|\xi|} \quad (\xi \leqslant \xi_0), \quad u_0(\xi) \geqslant \alpha\pi/3 \quad (\xi \geqslant \xi_0) \tag{8}$$

From (5) and (7) it follows that $0 \leqslant u_0(\xi) + u(\xi) \leqslant \pi - \delta$ when $\xi \in (-\infty, \infty)$ and

$$u_0 + u \geqslant \sin(u_0 + u) \geqslant \pi^{-1}\sin\delta u_0 \geqslant 0 \tag{9}$$

Using (6), (8) and (9) we obtain from (3)

$$0 \leqslant G(\xi) \leqslant 2\lambda/\pi \quad (|\xi| < \infty) \tag{10}$$

$$G(\xi) \leqslant \pi(\alpha\sin\delta\xi)^{-1} \quad (\xi \geqslant \xi_0) \tag{11}$$
$$G(\xi) \leqslant 8\alpha\pi^{-1}e^{-|\xi|} + 2\lambda\pi^{-1}C_1 e^{-\beta|\xi|} \quad (\xi \leqslant \xi_0)$$

Applying the operator D to (10) and remembering that $D(1) = \pi/2$ we obtain $0 \leqslant F(\xi) \leqslant \lambda$ and this, together with (4), yields

$$0 \leqslant F(\xi) \leqslant (1-\alpha)\pi - \delta \quad (|\xi| < \infty) \tag{12}$$

Below we shall use the following estimates:

$$D_1(e^{-\nu|\xi|}) \leqslant \frac{\pi}{2} f(\nu)e^{-\nu|\xi|} \quad (|\xi| < \infty) \tag{13}$$

$$D_2(1/\xi) \leqslant N\Phi(\xi) \quad (|\xi| < \infty) \tag{14}$$
$$\Phi(\xi) = e^{-|\xi|} \quad (\xi \leqslant \xi_0), \quad \Phi(\xi) = 1/\xi \quad (\xi > \xi_0)$$

where $N > 0$, $f(\nu)$ is a continuous increasing function, $f(0) = 1$ and $f(1) = \infty$. The inequality (13) has been proved in [1]. To prove (14), we shall consider the function $r(\zeta) = \ln\ln(e^{\zeta} + 2) - \ln\ln 2$ continuous for $0 \leqslant \eta \leqslant \pi/2$, $|\xi| < \infty$. We denote

$$p_1(\xi) = \operatorname{Im} r\left(\xi + i\frac{\pi}{2}\right) = \operatorname{arctg}\frac{a}{b}$$

$$p_2(\xi) = \frac{d}{d\xi}\operatorname{Re} r\left(\xi + i\frac{\pi}{2}\right) = \frac{2(2a + be^{\xi})e^{\xi}}{(a^2 + b^2)(e^{2\xi} + 4)}$$

$$(a = 2\operatorname{arctg}(e^{\xi}/2), \quad b = \ln(e^{2\xi} + 4))$$

The following inequalities hold:

$$p_1(\xi) \leqslant M\Phi(\xi) \quad (|\xi| < \infty), \quad p_2(\xi) \geqslant m/\xi > 0 \quad (\xi \geqslant \xi_0) \tag{15}$$

Taking into account the fact that $r(-\infty + i\eta) = 0$, $\operatorname{Im} r(\xi) = 0$ $(|\xi| < \infty)$ we have $p_1(\xi) = D[p_2(\xi)] \geqslant D_2[p_2(\xi)]$ and this, together with (15), yields

$$M\Phi(\xi) \geqslant mD_2(1/\xi) \quad (|\xi| < \infty) \tag{16}$$

The estimate (14) with $N = M/m$ now follows from (16).

Majorizing $e^{-|\xi|}$ in (11) with the function $e^{-\beta|\xi|}$ and applying the estimates (13) and (14) we obtain, from (11),

$$F(\xi) \leqslant \pi N(\alpha \sin\delta)^{-1}\Phi(\xi) + \lambda f(\beta)e^{-\beta|\xi|}(C_1 + 4\alpha) \quad (|\xi| < \infty) \tag{17}$$

Using (17), the properties of $f(\nu)$ and the inequalities $\lambda < 1$, $0 < \beta < 1$, we can show that a sufficiently small β and a sufficiently large C_1, and hence a sufficiently large C_2, can be chosen such that the following inequalities will hold:

$$F(\xi) \leqslant C_1 e^{-\beta|\xi|} \quad (\xi \leqslant \xi_0), \quad F(\xi) \leqslant \frac{C_2}{\xi} \quad (\xi \geqslant \xi_0) \tag{18}$$

Comparing (5) and (6) with (12) and (18) we conclude, that for the values of C_1, C_2 and β chosen the operator T transforms $H(C_1, C_2, \beta)$ into itself. Using tne estimates obtained, we can also show the complete continuity of the contraction of the operator T on H on the norm of the space E (the proof is based on the known fact that the family of functions $F_n(\xi) = D[G_n(\xi)]$ is equicontinuous on any finite interval provided that the family $G_n(\xi)$ is uniformly bounded for $|\xi| < \infty$).

From the Schauder principle it follows that when the conditions (4) hold, the equation $u = T(u)$, equivalent to the initial hydrodynamic problem, has at leastone solution $u(\xi) \in H$. Using the methods of [1, 3] we can extend the theorem of existence proved above to the case of a channel with a curvilinear bottom sloping without bounds.

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